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Nonnegative Solutions of a Nonlinear Recurrence

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Orthonormal polynomials with weight $|\tau|^\rho \exp(-\tau^4)$ have leading coefficients with recurrence properties which motivate the more general equations $\xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = \gamma_m^2$, $m = 1, 2,\ldots$, where ξ_0 is a fixed nonnegative value and $\gamma_1, \gamma_2,\ldots$ are positive constants. For this broader problem, the existence of a nonnegative solution is proved and criteria are found for its uniqueness. Then, for the motivating problem, an asymptotic expansion of its unique nonnegative solution is obtained and a fast computational algorithm, with error estimates, is given.

1. INTRODUCTION

Given $\rho > -1$ and $r = 1, 2,\ldots,$ let $w(\tau) = |\tau|^\rho \exp(-|\tau|^r)$, where $-\infty < \tau < +\infty$. Then the weight function $w(\tau)$ defines unique orthonormal polynomials $p_0(\tau)$, $p_1(\tau)$, $p_2(\tau)$,... such that

$$\int_{-\infty}^{\infty} p_m(\tau)\, p_n(\tau)\, w(\tau)\, d\tau = \delta_{mn} \tag{1.1}$$

and $p_m(\tau) = \pi_m \tau^m +$ lower terms, where the coefficients $\pi_m > 0$. We set $p_{-1}(\tau) = 0$, $\pi_{-1} = 0$. Freud [3–5] and Nevai [8] have studied the ratios π_{m-1}/π_m because these determine the polynomials $p_m(\tau)$. When $r = 2$, Freud shows

$$(\pi_{m-1}/\pi_m)^2 = (\tfrac{1}{4})\,[2m + \rho - \rho(-1)^m], \qquad m = 0, 1, 2,\ldots. \tag{1.2}$$

When $r = 4$, he finds that these ratios obey a nonlinear recurrence relation. Indeed, if $\xi_m = (\pi_{m-1}/\pi_m)^2$, then $\xi_0 = 0$ and all higher $\xi_m > 0$, while

$$\xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = (\tfrac{1}{8})\,[2m + \rho - \rho(-1)^m], \qquad m = 1, 2,\ldots. \tag{1.3}$$

Freud [5] shows that $\lim_m (12/m)^{1/2}\, \xi_m = 1$.

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Here we extend and sharpen these results. Taking any positive sequence $(\gamma_1, \gamma_2, \ldots)$ we fix $\xi_0 \geqslant 0$ and study the real sequences $(\xi_1, \xi_2, \ldots)$ such that

$$\xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = \gamma_m^2, \qquad m = 1, 2, \ldots. \tag{1.4}$$

Since the value ξ_0 is already given, each real ξ_1 inductively determines an infinite sequence $(\xi_1, \xi_2, \ldots)$ unless some component ξ_m is precisely zero—and this last condition excludes just countably many ξ_1. If $\gamma_m = \gamma$ and $\xi_0 \neq 0$ then (1.4) has a period-4 solution, namely,

$$(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_0, \gamma^2/\xi_0, \ -\gamma^2/\xi_0, \ -\xi_0, \xi_0, \ldots), \tag{1.5}$$

and this sequence has some negative terms; but our prime concern will be *nonnegative* ξ_m. Unless we state otherwise, hereafter the γ_m are positive and the ξ_m are nonnegative.

Other sequences give useful illustrations. If $\gamma_m = \beta m + \gamma$, where $\beta \geqslant 0$ and $\gamma = \xi_0\sqrt{3} > 0$, then the following satisfies (1.4):

$$\xi_m = (\beta m + \gamma)/\sqrt{3} \qquad \text{for } m = 1, 2, \ldots. \tag{1.6}$$

When $\beta = 0$, these constants γ_m include the previous example, but the values (1.6) exceed 0; so these ξ_m furnish a different solution. Yet another solution provides a later counterexample. If $\gamma_m = \gamma(\sigma^m - \sigma^{-m})$, where $\sigma > 1$ and $\gamma > \xi_0 = 0$, then the following satisfies (1.4):

$$\xi_m = \gamma(\sigma + 1 + \sigma^{-1})^{-1/2}(\sigma^m - \sigma^{-m}) \qquad \text{for } m = 1, 2, \ldots. \tag{1.7}$$

Further notation simplifies this work. The set R^n (resp. R^∞) of all real n-tuples (resp. real infinite sequences) is a vector space under componentwise addition and real scalar multiplication. The scalar 0, with no ambiguity, will denote the zero vector in either space. Typical elements x, y of these spaces will have respective components ξ_m, η_m for positive indices m; the scalars ξ_0, η_0 with subscript zero will *not* be components of the associated vectors. Given any $x = (\xi_1, \xi_2, \ldots)$ and $y = (\eta_1, \eta_2, \ldots)$, write $x \leqslant y$ (resp. $x < y$) if all $\xi_m \leqslant \eta_m$ (resp. $\xi_m < \eta_m$). Call x *nonnegative* when $0 \leqslant x$; call x *positive* when $0 < x$.

The finite-dimensional space R^n will use the standard l^∞ norm:

$$\|x\| = \max\{|\xi_m| : m = 1, \ldots, n\}. \tag{1.8}$$

More generally, let $0 < a = (\alpha_1, \alpha_2, \ldots) \in R^\infty$ and let $0 < b = (\beta_1, \beta_2, \ldots) \in R^\infty$. Then the set R^∞, with the norm

$$\|x\|_a = \sup\{|\xi_m/\alpha_m| : m = 1, 2, \ldots\} \tag{1.9}$$

is a complete metric space with distance $d(x, y) = \|x - y\|_a$. (This definition allows infinite distances, but these cause no difficulties.) If $\|\cdot\|_b$-convergence implies $\|\cdot\|_a$-convergence, then $\|b\|_a < +\infty$; however, $\|x\|_a \leqslant \|x\|_b \|b\|_a$, and this proves the converse. Thus a and b define the same topology (so that a and b yield *equivalent* norms) if and only if $\|a\|_b, \|b\|_a < +\infty$.

This work fixes positive $c = (\gamma_1, \gamma_2, ...) \in R^\infty$ and $\xi_0 \geqslant 0$; it seeks nonnegative $x = (\xi_1, \xi_2, ...) \in R^\infty$ where x satisfies (1.4). Section 2 obtains some basic results for an auxiliary linear recurrence with variable coefficients. Section 3 discusses the truncated problem of finite sequences $\xi_1, ..., \xi_n$ with fixed ξ_0, ξ_{n+1}. Sections 4, 5, 6, for infinite sequences, using an equivalent formulation as a fixed-point problem, prove the existence of nonnegative solutions, and give several criteria for uniqueness. Conversely, an argument in Section 6 generates multiple nonnegative solutions for some vectors c. Also, the stronger hypotheses of Section 5 yield important results for norm-convergence. Section 7, on sequence computation, shows the instability of forward iteration and gives a stable algorithm, with error estimates. Sections 8 and 9, for the original problem (1.3), find an asymptotic expansion of the unique nonnegative solution, and report computational experience with a still more refined algorithm.

2. Linear Recurrence

Given any complex sequence $(\omega_1, \omega_2, ...)$, consider the complex sequence $(\zeta_0, \zeta_1, \zeta_2, ...)$ such that

$$\zeta_{m+1} - \omega_m \cdot \zeta_m + \zeta_{m-1} = 0 \tag{2.1}$$

when $m = 1, 2, ...$. This section, for certain recurrences (2.1), extends a well-known theorem of Poincaré (Montel [7, Chap. 5]) and collects auxiliary results for later use. If $(\zeta_0, \zeta_1) = (1, 0)$ or $(0, 1)$, respectively, then $\omega_1, ..., \omega_m$ determine the value ζ_{m+1}, whence ζ_{m+1} is some function $B_m(\omega_1, ..., \omega_m)$ or $C_m(\omega_1, ..., \omega_m)$, and $B_0 = 0$, $C_0 = 1$. Then the linearity of (2.1) gives the solution for any (ζ_0, ζ_1):

$$\zeta_{m+1} = \zeta_0 B_m(\omega_1, ..., \omega_m) + \zeta_1 C_m(\omega_1, ..., \omega_m). \tag{2.2}$$

If $\omega_1 = \cdots = \omega_m = \omega$ then we adopt a simpler notation: $B_m(\omega_1, ..., \omega_m) = B_m(\omega)$ and $C_m(\omega_1, ..., \omega_m) = C_m(\omega)$. Also, $U_m(\tau)$, for each integer m, is the m'th Chebyshev polynomial of the second kind (Abramowitz and Stegun [1, (22.2.5)]).

Lemma 2.1. *If $m = 0, 1, 2, ...$ then $C_m(\omega) = U_m(\omega/2)$. Indeed, $C_m(2) = m + 1$ and $C_m(-2) = (-1)^m (m + 1)$. If $\omega \neq \pm 2$ and σ is either complex* *number such that* $\sigma + \sigma^{-1} = \omega$, *then* $C_m(\omega) = (\sigma^{m+1} - \sigma^{-m-1})/(\sigma - \sigma^{-1})$. *Also*, $B_m(\omega) = -C_{m-1}(\omega)$ *for* $m = 0, 1, 2, \ldots$, *where these explicit forms define* $C_m(\omega)$ *for* $m < 0$.

Proof. The explicit forms $C_m(\omega)$ and $-C_{m-1}(\omega)$ satisfy the recurrence (2.1). The resulting expressions $B_m(\omega)$ and $C_m(\omega)$ reproduce the stated ζ_0 and ζ_1. If $\omega = 2 \cos\theta$ and $0 < \theta < \pi$, then $\sigma = \exp(\pm i\theta)$ and (Abramowitz and Stegun [1, Eq. (22.3.16)]):

$$C_m(\omega) = \sin(m+1)\theta \sin\theta = U_m(\cos\theta) = U_m(\omega/2). \tag{2.3}$$

Analytic continuation admits all other complex ω.

Now let I_n be the $n \times n$ identity matrix, where $n = 1, 2, \ldots$, and let E_n be the $n \times n$ matrix (ε_{ij}) such that $\varepsilon_{ij} = 1$ when $|i - j| = 1$ but otherwise $\varepsilon_{ij} = 0$. If $\theta_{mn} = m\pi/(n+1)$, where $m = 1, \ldots, n$, then

$$[E_n \cdot (\sin\theta_{mn}, \ldots, \sin n\theta_{mn})^{\text{transpose}}]_k = 2\cos\theta_{mn} \cdot \sin k\theta_{mn} \tag{2.4}$$

for $k = 1, \ldots, n$. Thus E_n has the n distinct eigenvalues $2\cos\theta_{mn}$, and these have the corresponding nonzero eigenvectors $(\sin\theta_{mn}, \ldots, \sin n\theta_{mn})^{\text{transpose}}$.

LEMMA 2.2. *If* $n = 0, 1, 2, \ldots$ *then*

$$C_n(\omega_1, \ldots, \omega_n) = \det(E_n + \operatorname{diag}(\omega_1, \ldots, \omega_n)). \tag{2.5}$$

If all $\omega_m \geqslant 2$, *where* $m = 1, \ldots, n$, *then* $E_n + \operatorname{diag}(\omega_1, \ldots, \omega_n)$ *is a positive definite matrix, and*

$$0 < C_n(\min(\omega_1, \ldots, \omega_n)) \leqslant C_n(\omega_1, \ldots, \omega_n) \leqslant C_n(\max(\omega_1, \ldots, \omega_n)). \tag{2.6}$$

If also $\omega'_m \geqslant \omega_m$ *and some* $\omega'_k > \omega_k$, *where* $1 \leqslant k \leqslant n$, *then*

$$C_n(\omega_1, \ldots, \omega_n) < C_n(\omega'_1, \ldots, \omega'_n). \tag{2.7}$$

Proof. A bottom-row expansion of $\det(E_n + \operatorname{diag}(\omega_1, \ldots, \omega_n))$ obtains the recurrence (2.1) for these determinants. (Any such expansion by minors assigns the value 1 to a zero-order determinant.) But inspection yields (2.5) when $n = 0, 1, 2$; so induction proves (2.5) when $n = 0, 1, \ldots$. The matrix $E_n + 2I_n$ is positive definite, because the principal minors have the form $C_m(2) = m + 1 > 0$. The matrix $\operatorname{diag}(\omega_1 - 2, \ldots, \omega_n - 2)$ is nonnegative definite whenever all $\omega_m \geqslant 2$. These matrices have positive definite sum $E_n + \operatorname{diag}(\omega_1, \ldots, \omega_n)$, which has strictly positive determinant $C_n(\omega_1, \ldots, \omega_n)$. But expansion via the mth row gives

$$\partial C_n(\omega_1, \ldots, \omega_n)/\partial\omega_m = C_{m-1}(\omega_1, \ldots, \omega_{m-1}) \cdot C_{n-m}(\omega_{m+1}, \ldots, \omega_n) > 0; \tag{2.8}$$

and this fact, for all variables ω_m, yields (2.7). Finally, (2.7) implies (2.6).

THEOREM 2.3. *If* $2 \leqslant \omega_1, \omega_2,\ldots$ *and* $0 < \varepsilon$ *such that* $2 + \varepsilon < \omega_- = \liminf_m \omega_m < +\infty$ *and* $\omega_+ = \limsup_m \omega_m \leqslant +\infty$, *then this* ε *determines constants* $\gamma_-(\varepsilon)$, $\gamma_+(\varepsilon) > 0$ *such that*

$$\gamma_-(\varepsilon)\, C_m(\omega_- - \varepsilon) \leqslant C_m(\omega_1,\ldots, \omega_m) \leqslant \gamma_+(\varepsilon)\, C_m(\omega_+ + \varepsilon), \qquad m = 0, 1, 2,\ldots. \tag{2.9}$$

Proof. Any positive ε fixes an integer n such that $\omega_m \geqslant \omega_- - \varepsilon$ whenever $m > n$. If $\omega'_m = \omega_m$ for $m \leqslant n$, while $\omega'_m = \omega_- - \varepsilon$ *for* $m > n$, then

$$m + 1 = C_m(2) \leqslant C_m(\omega'_1,\ldots, \omega'_m) \leqslant C_m(\omega_1,\ldots, \omega_m) \tag{2.10}$$

for $m = 0, 1, 2,\ldots$. Choose the root $\sigma > 1$ satisfying the equation $\sigma + \sigma^{-1} = \omega_- - \varepsilon$. If $m > n$ then Lemma 2.1 implies constants α, β such that

$$m + 1 \leqslant C_m(\omega'_1,\ldots, \omega'_m) = \alpha\sigma^{m-n} + \beta\sigma^{n-m}. \tag{2.11}$$

(To prove this, replace m by $m - n$.) Moreover, $\alpha > 0$, since $m + 1$ may be larger than $|\beta|$. Thus $C_m(\omega'_1,\ldots, \omega'_m)/C_m(\omega_- - \varepsilon) > 0$ when $m = 0, 1, 2,\ldots,$ and this ratio has a positive limit as $m \to \infty$, whence

$$C_m(\omega_1,\ldots, \omega_m)/C_m(\omega_- - \varepsilon) \geqslant \inf_m C_m(\omega'_1,\ldots, \omega'_m)/C_m(\omega_- - \varepsilon) = \gamma_-(\varepsilon) > 0, \tag{2.12}$$

when $m = 0, 1, 2,\ldots$. Similar arguments find $\gamma_+(\varepsilon)$ unless $\omega_+ = +\infty$, but then $C_m(\omega_+ + \varepsilon) = +\infty$ whenever $m > 0$.

Remark. One easily finds examples where $\gamma_-(\varepsilon) \to 0$ or $\gamma_+(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

3. FINITE SEQUENCES

Here, keeping the positive constants $\gamma_1, \gamma_2,\ldots,$ we specify a positive integer n, and, fixing real numbers ξ_0, ξ_{n+1}, we seek real values $\xi_1,\ldots, \xi_n$ such that

$$\xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = \gamma_m^2 \qquad \text{for } m = 1,\ldots, n. \tag{3.1}$$

First, we restate the problem. Given $x = (\xi_1,\ldots, \xi_n) \in R^n$ and $\xi_1,\ldots, \xi_n \neq 0$, define

$$F(x) = -\sum_{m=1}^{n} \gamma_m^2 \log|\xi_m| + (\tfrac{1}{2}) \sum_{m=0}^{n+1} \xi_m^2 + \sum_{m=0}^{n} \xi_m\xi_{m+1}; \tag{3.2}$$

then $F'(x) = (\partial F/\partial\xi_1,\ldots, \partial F/\partial\xi_n)$, where

$$\partial F/\partial\xi_m = -\gamma_m^2\xi_m^{-1} + \xi_{m-1} + \xi_m + \xi_{m+1} \qquad \text{for } m = 1,\ldots, n. \tag{3.3}$$

Clearly, x satisfies (3.1) if and only if $F'(x)=0$, because necessarily $\xi_1,\dots,\xi_n \neq 0$ when x satisfies either of these conditions. Hence the solutions of (3.1) are precisely the stationary points of $F(x)$.

Truncations of sequence (1.5) yield solutions for equal γ_m having negative values for some ξ_m. (Appendix B gives further such solutions.) Nevertheless, here we suppose nonnegative ξ_0, ξ_{n+1}, and again we admit only nonnegative $\xi_1,\dots,\xi_n$. But if $x \geqslant 0$, then, by (1.8),

$$F(x) \geqslant -\left(\sum_{m=1}^{n} \gamma_m^2\right) \log \|x\| + (\tfrac{1}{2})\|x\|^2; \tag{3.4}$$

while if $k=1,\dots,n$, then, for each k,

$$F(x) \geqslant -\gamma_k^2 \log |\xi_k| - \left(\sum_{m \neq k} \gamma_m^2\right) \log \|x\|. \tag{3.5}$$

Hence $\{x \in R^n : x \geqslant 0; \sigma \geqslant F(x)\}$, for any real σ, is bounded in the norm (1.8); and this level set, via this fact, is bounded away from the coordinate hyperplanes. Also, the set is closed, whence it is compact. Thus $F(x)$, on the orthant $\{x \in R^n : x \geqslant 0\}$, achieves its minimum at some strictly positive x^*. This x^* is a nonnegative solution; our next theorem shows that no other x is a nonnegative solution.

LEMMA 3.1. *If x satisfies* (3.1), *where $\xi_0, \xi_1,\dots,\xi_{n+1} \geqslant 0$, then either $0 < \xi_m < \gamma_m$ for $m=1,\dots,n$ or, specifically, $n=1$ and $(\xi_0, \xi_1, \xi_2) = (0, \gamma_1, 0)$.*

Proof. If nonnegative $\xi_1,\dots,\xi_n$ satisfy (3.1) then each exceeds zero. If $n=1$ and $\xi_0=\xi_2=0$, then $\xi_1=\gamma_1$ since $\xi_1>0$. Otherwise $\xi_{m-1}+\xi_{m+1}>0$ for $m=1,\dots,n$; whence $\xi_m^2 < \xi_m(\xi_{m-1}+\xi_m+\xi_{m+1}) = \gamma_m^2$ and $\xi_m < \gamma_m$.

THEOREM 3.2. *If $\xi_0, \xi_{n+1} \geqslant 0$ then $F(x)$ has a unique stationary point x^* in $\{x \in R^n : x \geqslant 0\}$, and this is the unique nonnegative solution of* (3.1). *Moreover, $F(x)$, on this orthant, achieves its minimum at x^*.*

Proof. Let the set $K = \{x \in R^n : 0 \leqslant \xi_m \leqslant \gamma_m, m=1,\dots,n\}$. Then the nonnegative stationary points of $F(x)$ are just the nonnegative solutions of (3.1), and any such solutions, by Lemma 3.1, are necessarily elements of the set K. By definition, the Hessian $F''(x)$ is the $n \times n$ matrix $(\partial^2 F/\partial\xi_i \partial\xi_j)$; by (3.3),

$$F''(x) = E_n + \operatorname{diag}(1+\gamma_1^2\xi_1^{-2},\dots,1+\gamma_n^2\xi_n^{-2}). \tag{3.6}$$

Here Section 2 defines the matrix E_n. If x has domain K then, by definition, $1+\gamma_m^2\xi_m^{-2} \geqslant 2$ when $m=1,\dots,n$; so, by Lemma 2.2, $F''(x)$ is positive definite and $F(x)$ is strictly convex (Ortega and Rheinboldt [10, p. 87]). Thus $F(x)$, on the domain K, can have no stationary point but a unique minimum. However, $F(x)$ on $\{x \in R^n: x \geqslant 0\}$ has a global minimum by our previous remarks.

COROLLARY 3.3. *For $n(1)$, $n(2)$ any nonnegative integers and $k = 1, 2$, let $\xi_0^{(k)},..., \xi_{n(k)+1}^{(k)} \geqslant 0$. For $\gamma_1, \gamma_2,...$ any positive reals and $m = 1,..., n(k)$, let*

$$\xi_m^{(k)}(\xi_{m-1}^{(k)} + \xi_m^{(k)} + \xi_{m+1}^{(k)}) = \gamma_m^2. \tag{3.7}$$

If $\xi_i^{(1)} = \xi_i^{(2)}$ for any distinct subscripts p, q, then $\xi_i^{(1)} = \xi_i^{(2)}$ for all common indices i.

Proof. Assuming $p < q$, replace m by $m - p$. Setting $n = q - p - 1$, use Theorem 3.2 if $n > 0$. Then $\xi_i^{(1)} = \xi_i^{(2)}$ when the new index $i = 0,..., n + 1$ or the old index $i = p,..., q$. Forward or backward recurrence determines uniquely all other $\xi_i^{(k)}$.

COROLLARY 3.4. *If ξ_0 is a constant and $\xi_1,..., \xi_n$ obey (3.1), then $\xi_{n+1}(\xi_1) = q_{n+1}(\xi_1)/r_{n+1}(\xi_1)$, where q_{n+1}, r_{n+1} are real polynomials. Indeed,*

$$\frac{q_{m+1}}{r_{m+1}} = \frac{\gamma_m^2 r_m^2 r_{m-1} - q_m^2 r_{m-1} - q_m r_m q_{m-1}}{q_m r_m r_{m-1}}, \qquad m = 1,..., n, \tag{3.8}$$

where $q_0(\xi)$ is the constant ξ_0, $q_1(\xi) = \xi$, $r_0(\xi) = r_1(\xi) = 1$. Moreover, ξ_{n+1} has an open domain S_{n+1}, satisfying $[0, +\infty) \subset S_{n+1}$, whereon ξ_{n+1} uniquely determines ξ_1 provided $\xi_0,..., \xi_n \geqslant 0$. The function $\xi_1(\xi_{n+1})$ for each n is monotone and bicontinuous on this domain.

Proof. Direct use of (3.1) proves everything through (3.8). By Corollary 3.3, if $S = \{\xi_1: 0 \leqslant \xi_0,..., \xi_{n+1}\}$ then $\xi_{n+1}(\xi_1)$, on S, has a well-defined inverse function, whence $\xi_{n+1}(\xi_1)$, on S, has no local extrema. Specifically, this function has no minimum where $\xi_{n+1}(\xi_1) = 0$. Hence $\xi_{n+1}(\xi_1)$ is continuous and monotonic on some larger open domain S' including S, and its inverse is continuous and monotonic on an open domain S_{n+1} including $[0, +\infty)$.

4. INFINITE SEQUENCES

Here we treat the infinite system (1.4): demanding nonnegative solutions, we prove a general existence theorem and obtain a uniqueness criterion. First, we state an equivalent fixed-point problem, using an auxiliary function g. Specifically, we let

$$\sigma = g(\tau) = -\tau + (1 + \tau^2)^{1/2}, \tag{4.1}$$

where $-\infty < \tau < +\infty$; equivalently, we have

$$1 = \sigma^2 + 2\sigma\tau \qquad \text{and} \qquad \sigma > 0. \tag{4.2}$$

Thus $g'(\tau) = d\sigma/d\tau = -2/(1+\sigma^{-2})$, and $g(0) = -g'(0) = 1$. Hence $g(\tau)$ and $-g'(\tau)$ decrease from 1 to 0 as τ increases from 0 to $+\infty$. Fixing real $\xi_0 \geqslant 0$ and $0 < c = (\gamma_1, \gamma_2, \ldots) \in R^\infty$, define the map $T\colon R^\infty \to R^\infty$ via

$$(Tx)_m = \gamma_m \cdot g((\xi_{m-1} + \xi_{m+1})/2\gamma_m), \qquad m = 1, 2, \ldots. \tag{4.3}$$

Then (1.4), for each positive m, is a quadratic equation in ξ_m, and the nonnegative root of this is the right side of (4.3). Therefore the nonnegative solutions of (1.4) are precisely the fixed points of T. Our first lemmas concern this map T.

LEMMA 4.1. *If 0 is the zero vector then $T0 \leqslant c$; if x is any real vector then $0 < Tx$. If $x \leqslant y$ (resp. $x < y$) then $Ty \leqslant Tx$ (resp. $Ty < Tx$) and $T^2x \leqslant T^2y$ (resp. $T^2x < T^2y$). If $\xi_0 = \eta_0$ and $x, y \in R^\infty$, while*

$$\tau_m \leqslant (1/2\gamma_m)\min(\xi_{m-1} + \xi_{m+1}, \eta_{m-1} + \eta_{m+1}) \tag{4.4}$$

for $m = 1, 2, \ldots$, then

$$|(Ty)_m - (Tx)_m| \leqslant |g'(\tau_m)/2| \cdot \{|\eta_{m-1} - \xi_{m-1}| + |\eta_{m+1} - \xi_{m+1}|\}$$
$$\textit{for} \quad m = 1, 2, \ldots. \tag{4.5}$$

Proof. Clearly $(T0)_1 = \gamma_1 g(\xi_0/2\gamma_1) \leqslant \gamma_1$, since $\xi_0 \geqslant 0$; and $(T0)_m = \gamma_m$ when $m \geqslant 2$. Also, any $(Tx)_m > 0$ because g is strictly positive, and T reverses inequalities because g is strictly decreasing. Finally, the mean value theorem asserts

$$(Ty)_m - (Tx)_m = (\tfrac{1}{2})\, g'(\tau_m)\,[(\eta_{m-1} + \eta_{m+1}) - (\xi_{m-1} + \xi_{m+1})], \tag{4.6}$$

where the interval containing τ_m has bounds $(\xi_{m-1} + \xi_{m+1})/2\gamma_m$ and $(\eta_{m-1} + \eta_{m+1})/2\gamma_m$. But we increase $|g'(\tau_m)|$ when we decrease τ_m.

LEMMA 4.2. *If $k = 0, 1, 2, \ldots$, then*

$$0 = T^0 0 \leqslant T^{2k}0 < T^{2k+2}0 < T^{2k+3}0 < T^{2k+1}0 \leqslant T0. \tag{4.7}$$

Thus, componentwise, $T^{2k}0$ has monotone increasing limit x^- and, similarly, $T^{2k+1}0$ has monotone decreasing limit x^+ as $k \to \infty$. Also, $Tx^- = x^+$ and $Tx^+ = x^-$, while

$$Tc < x^- \leqslant x^+ < c. \tag{4.8}$$

If T has fixed point x^*, so that $Tx^* = x^*$, then $x^- \leqslant x^* \leqslant x^+$.

Proof. Lemma 4.1 shows, first, that $0 < T^k 0$ if $k = 1, 2, \ldots$, whence $T^{k+1}0 < T0$. Specializing these facts gives $0 < T^2 0 < T0$; then using T gives $T^2 0 < T^3 0 < T0$. Together, these are (4.7) when $k = 0$. Repeatedly using T^2 inductively completes (4.7). If j is any positive integer, then the increasing sequence $T^0 0$, $T^2 0$, $T^4 0, \ldots$ has upper bound $T^{2j+1}0$, and the decreasing sequence $T0$, $T^3 0$, $T^5 0, \ldots$ has lower bound $T^{2j}0$; so these sequences have componentwise limits x^-, x^+. Thus $Tc \leqslant T^2 0 < x^- \leqslant T^{2j+1}0$, while $T^{2j}0 \leqslant x^+ < T0 \leqslant c$, and these yield (4.8) when $j \to \infty$. But g is continuous, whence

$$\begin{aligned}(Tx^-)_m &= \gamma_m \cdot g(\lim_k((T^{2k}0)_{m-1} + (T^{2k}0)_{m+1})/2\gamma_m) \\ &= \gamma_m \cdot \lim_k g(((T^{2k}0)_{m-1} + (T^{2k}0)_{m+1})/2\gamma_m) \\ &= \lim_k (T^{2k+1}0)_m = (x^+)_m. \end{aligned} \tag{4.9}$$

Therefore $Tx^- = x^+$; likewise $Tx^+ = x^-$. If $Tx^* = x^*$ then $0 < x^*$; so $T^{2k}0 < T^{2k}x^* = x^* = T^{2k+1}x^* < T^{2k+1}0$, where $k = 0, 1, 2, \ldots$. Let $k \to \infty$ to find $x^- \leqslant x^* \leqslant x^+$.

Remark. The sequences x^- and x^+ underlie many later results.

THEOREM 4.3. *If $\xi_0 \geqslant 0$ and $c > 0$ then T has a fixed point, whence* (1.4) *has a nonnegative solution. Either $x^- = x^+$ and this solution is unique, or $x^- < x^+$ and this solution is not unique. Specifically, it is unique when* $\inf_m \gamma_m / m = 0$.

Proof. Given $x^- = (\xi_1^-, \xi_2^-, \ldots)$ and $x^+ = (\xi_1^+, \xi_2^+, \ldots)$, define

$$x^{\pm} = (\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \ldots) \qquad \text{and} \qquad x^{\mp} = (\xi_1^-, \xi_2^+, \xi_3^-, \xi_4^+, \ldots). \tag{4.10}$$

Clearly $Tx^{\pm} = x^{\pm}$ and $Tx^{\mp} = x^{\mp}$, because $Tx^- = x^+$ and $Tx^+ = x^-$. If solution x^* is unique then $x^{\pm} = x^{\mp}$, whence $x^- = x^+$; conversely, solution x^* is unique whenever $x^- = x^+$, since $x^- \leqslant x^* \leqslant x^+$. But $x^- = x^+$, by Corollary 3.3, if $\xi_m^- = \xi_m^+$ for any positive m. If $\zeta_m = \xi_m^+ - \xi_m^-$ for $m = 0, 1, 2, \ldots$, then $\zeta_0 = 0$, since $\xi_0^+ = \xi_0^- = \xi_0$. Also,

$$\begin{aligned}\zeta_{m-1} + \zeta_{m+1} &= (\xi_{m-1}^+ + \xi_{m+1}^+) - (\xi_{m-1}^- + \xi_{m+1}^-) \\ &= (\gamma_m^2/\xi_m^- - \xi_m^-) - (\gamma_m^2/\xi_m^+ - \xi_m^+) = \omega_m \cdot \zeta_m \end{aligned} \tag{4.11}$$

by (4.2) and Lemma 4.2, where $\omega_m = 1 + \gamma_m^2/\xi_m^+ \xi_m^- > 2$ by (4.8). However, $\zeta_1, \zeta_2, \ldots > 0$ for multiple solutions, whence

$$\gamma_m > \zeta_m = \zeta_1 \cdot C_{m-1}(\omega_1, \ldots, \omega_{m-1}) > \zeta_1 C_{m-1}(2) = m\zeta_1 \tag{4.12}$$

by (2.2), (2.7), and (4.8), where $m = 1, 2, \ldots$; thus $\inf_m \gamma_m / m \geqslant \zeta_1 > 0$.

5. Norm-Convergence

Here we add a hypothesis on the given vector c, and we treat convergence in the sequence norms (1.9). Choosing any $\gamma_0 \geqslant 0$, define $\phi_c(\gamma_0) = \inf S(c)$ and $\psi_c(\gamma_0) = \sup S(c)$, where $S(c) = \{(\gamma_{m-1} + \gamma_{m+1})/2\gamma_m \colon m = 1, 2,...\}$. Previously, we required only $0 < c \in R^\infty$, and thus $0 \leqslant \phi_c \leqslant \psi_c \leqslant +\infty$. Hereafter, we assume also

$$0 < \inf\{\gamma_{m+1}/\gamma_m \colon m = 1, 2,...\} \leqslant \sup\{\gamma_{m+1}/\gamma_m \colon m = 1, 2,...\} < +\infty, \quad (5.1)$$

whence now $0 < \phi_c \leqslant \psi_c < +\infty$. Conversely, the finiteness of ψ_c implies (5.1), independently of γ_0. Also, $\phi_c(\gamma_0)$ and $\psi_c(\gamma_0)$ are continuous, and $\phi_c(\gamma_0)$ is bounded, while $\psi_c(\gamma_0) \to +\infty$ as $\gamma_0 \to +\infty$. If $0 \leqslant \beta_0$ and $0 < b \in R^\infty$, then $\phi_b(\beta_0)$ and $\psi_b(\beta_0)$ have similar definitions and properties. Moreover, b is a concave sequence when $\psi_b \leqslant 1$.

If $0 \leqslant \kappa < \lambda \leqslant 1$ then $0 \leqslant \kappa\phi_c < \lambda\psi_c < +\infty$, whence $0 < g(\lambda\psi_c) < g(\kappa\phi_c) \leqslant 1$. Taking $\kappa_0 = 0$, define

$$\lambda_j = g(\kappa_j \phi_c), \qquad \kappa_{j+1} = g(\lambda_j \psi_c), \qquad j = 0, 1, 2,.... \quad (5.2)$$

Then $\lambda_0 = 1$ and $\kappa_1 = g(\psi_c)$, so that $0 = \kappa_0 < \kappa_1 < \lambda_0 = 1$ and, inductively,

$$0 = \kappa_0 < \cdots < \kappa_j < \kappa_{j+1} < \lambda_{j+1} < \lambda_j < \cdots < \lambda_0 = 1. \quad (5.3)$$

Hence $\kappa_j \uparrow \kappa_*$ and $\lambda_j \downarrow \lambda_*$ as $j \to \infty$, where $0 < \kappa_* \leqslant \lambda_* < 1$. Also, by continuity, $\kappa_* = g(\lambda_* \psi_c)$ and $\lambda_* = g(\kappa_* \phi_c)$, whence, by (4.2),

$$1 = \kappa_*^2 + 2\kappa_* \lambda_* \psi_c \qquad \text{and} \qquad 1 = \lambda_*^2 + 2\kappa_* \lambda_* \phi_c. \quad (5.4)$$

Eliminating unity, we find λ_*/κ_*; substituting this, we find κ_*:

$$\lambda_*/\kappa_* = (\psi_c - \phi_c) + [1 + (\psi_c - \phi_c)^2]^{1/2} \geqslant 1, \quad (5.5)$$

$$\kappa_*^{-2} = 1 + 2\psi_c(\psi_c - \phi_c) + 2\psi_c[1 + (\psi_c - \phi_c)^2]^{1/2}. \quad (5.6)$$

Thus $\kappa_*(\gamma_0)$ and $\lambda_*(\gamma_0)$ are continuous functions. If $\gamma_0 \to +\infty$, then $\psi_c \to +\infty$ while ϕ_c has an upper bound; but $(1 - \lambda_*^2)/\phi_c = (1 - \kappa_*^2)/\psi_c$, whence $\lambda_* \to 1$ and $\gamma_0 \lambda_* \to +\infty$. Also, $\gamma_0 \kappa_* = \gamma_0 \lambda_* = 0$ when $\gamma_0 = 0$. Previously, we have assumed a nonnegative ξ_0; hereafter, we can and will choose a nonnegative γ_0 such that

$$\gamma_0 \kappa_*(\gamma_0) \leqslant \xi_0 \leqslant \gamma_0 \lambda_*(\gamma_0). \quad (5.7)$$

Given $y, z \in R^\infty$, define $[y, z] = \{x \in R^\infty \colon y \leqslant x \leqslant z\}$. If $0 \leqslant \kappa \leqslant \kappa_* \leqslant \lambda_* \leqslant \lambda \leqslant 1$, and $x = (\xi_1, \xi_2,...) \in [\kappa c, \lambda c]$, then

$$\kappa\phi_c \leqslant \kappa(\gamma_{m-1} + \gamma_{m+1})/2\gamma_m \leqslant (\xi_{m-1} + \xi_{m+1})/2\gamma_m \leqslant \lambda(\gamma_{m-1} + \gamma_{m+1})/2\gamma_m \leqslant \lambda\psi_c, \quad (5.8)$$

whence $Tx \in [g(\lambda\psi_c)c, g(\kappa\phi_c)c]$. Therefore, if $j = 0, 1, 2, \ldots$, then, inductively,

$$
\begin{aligned}
T([\kappa_j c, \lambda_j c]) &\subset [\kappa_{j+1} c, \lambda_j c] \subset [\kappa_j c, \lambda_j c], \\
T([\kappa_{j+1} c, \lambda_j c]) &\subset [\kappa_{j+1} c, \lambda_{j+1} c] \subset [\kappa_{j+1} c, \lambda_j c].
\end{aligned}
\tag{5.9}
$$

But $0 < Tx$ if $x \in R^\infty$; so $0 < T^2 x < c$, or $T^2 x \in [\kappa_0 c, \lambda_0 c]$. Thus $T^{2j+2} x \in [\kappa_j c, \lambda_j c]$, where $j = 0, 1, 2, \ldots$. Hence $[\kappa_j c, \lambda_j c]$, invariant under T, must contain all fixed points of T; though $[\kappa_* c, \lambda_* c]$, by example (1.7), need contain no iterates $T^j 0$ for finite j (since $T^{2j} 0 = \kappa_j c$ and $T^{2j+1} 0 = \lambda_j c$). These new concepts yield some sharper results.

THEOREM 5.1. *If* $\inf_m \gamma_m/\sigma^m = 0$, *where* $\sigma + \sigma^{-1} = 1 + \lambda_*^{-2}$ *and* $\sigma > 1$, *then T has a unique fixed point.*

Proof. If $x^\pm$, $x^\mp$ are distinct fixed points then $\kappa_* c \leqslant x^- < x^+ \leqslant \lambda_* c$. If $\zeta_m = \xi_m^+ - \xi_m^-$, where $m = 0, 1, 2, \ldots$, then ζ_m satisfies (4.11), where $\omega_m = 1 + \gamma_m^2/\xi_m^+ \xi_m^- > 1 + \lambda_*^{-2} > 2$. But $\zeta_0 = 0$ and $\zeta_1 > 0$. Therefore $\inf_m \gamma_m/\sigma^m > 0$, because

$$
\begin{aligned}
\gamma_{m+1} > \zeta_{m+1} &= \zeta_1 C_m(\omega_1, \ldots, \omega_m) > \zeta_1 C_m(1 + \lambda_*^{-2}) \\
&= \zeta_1(\sigma^{m+1} - \sigma^{-m-1})/(\sigma - \sigma^{-1}).
\end{aligned}
\tag{5.10}
$$

LEMMA 5.2. *Let* $0 \leqslant \xi_0 = \eta_0$ *and* $\kappa_j c \leqslant x, y \in R^\infty$ (*and* $\gamma_0 \kappa_* \leqslant \xi_0 \leqslant \gamma_0 \lambda_*$), *where* $j \geqslant 0$. *Let* $0 \leqslant \beta_0$ *and* $0 < b = (\beta_1, \beta_2, \ldots) \in R^\infty$ (*where* ψ_b *or* $\|y - x\|_b$ *may be* $+\infty$); *then*

$$
\|Ty - Tx\|_b \leqslant \psi_b |g'(\kappa_j \phi_c)| \cdot \|y - x\|_b. \tag{5.11}
$$

Proof. If $m = 1, 2, \ldots$ then $\kappa_j \phi_c \leqslant (1/2\gamma_m) \min(\xi_{m-1} + \xi_{m+1}, \eta_{m-1} + \eta_{m+1})$, whence (4.5) shows that

$$
\begin{aligned}
&\frac{|(Ty)_m - (Tx)_m|}{\beta_m} \\
&\quad \leqslant \frac{|g'(\kappa_j \phi_c)|}{2} \left\{ \frac{\beta_{m-1}}{\beta_m} \frac{|\eta_{m-1} - \xi_{m-1}|}{\beta_{m-1}} + \frac{\beta_{m+1}}{\beta_m} \frac{|\eta_{m+1} - \xi_{m+1}|}{\beta_{m+1}} \right\}.
\end{aligned}
\tag{5.12}
$$

THEOREM 5.3. *Let* $0 \leqslant \beta_0$ *and* $0 < b \in R^\infty$ *such that* $\psi_b |g'(\kappa_* \phi_c)| < 1$. *Let* $0 \leqslant \xi_0$ *and* $0 \leqslant x^0 \in R^\infty$ *such that* $\|Tx^0 - x^0\|_b < +\infty$. *If* $K = \{x \in R^\infty : 0 \leqslant x;\ \|x - x^0\|_b < +\infty\}$ *then* $T(K) \subset K$ *and* $T|K$ *has a unique fixed point* x^*. *If* $y \in K$ *then* $\|T^j y - x^*\|_b \to 0$ *as* $j \to \infty$.

Proof. If $x \in K$ then $\kappa_0 c \leqslant x, x^0$, and Lemma 5.2 shows that

$$\begin{aligned} \| Tx - x^0 \|_b &\leqslant \| Tx - Tx^0 \|_b + \| Tx^0 - x^0 \|_b \\ &\leqslant \psi_b \| x - x^0 \|_b + \| Tx^0 - x^0 \|_b < +\infty. \end{aligned} \tag{5.13}$$

But $Tx \geqslant 0$; so $Tx \in K$. Also, $\psi_b |g'(\kappa_j \psi_0)| < 1$ for large enough j, and $K \cap [\kappa_j c, \lambda_j c]$ is a T-invariant set by (5.9). Hence T, on this set, is a contraction map by (5.11). If x^* is its unique fixed point, and $y \in K$, then $T^{2j+2} y \in K \cap [\kappa_j c, \lambda_j c]$ and $\lim_k \| T^{2j+2+k} y - x^* \|_b = 0$.

COROLLARY 5.4. *Let $0 \leqslant \beta_0$ and $0 < b \in R^\infty$. Let $\psi_b \leqslant 1$ (i.e., b is concave) and $\| \cdot \|_b$, $\| \cdot \|_c$ be equivalent norms. Then T has a unique fixed point x^* in R^∞, and $\lim_j \| T^j x - x^* \|_c = 0$ for any vector x.*

Proof. Here $\psi_b |g'(\kappa_* \phi_c)| \leqslant |g'(\kappa_* \phi_c)| < 1$ because $\kappa_* \phi_c > 0$. If $x^0 = 0$ then $0 < Tx^0 \leqslant c$; so $\| Tx^0 - x^0 \|_c \leqslant 1$ and $\| Tx^0 - x^0 \|_b < +\infty$. Thus $K = \{x \in R^\infty : 0 \leqslant x; \| x \|_c < +\infty\}$ and $T | K$ has a unique fixed point x^*. If $x \in R^\infty$ then $0 < Tx$ and $T^2 x \in K$, whence $\lim_j \| T^{j+2} x - x^* \|_c = \lim_j \| T^{j+2} x - x^* \|_b = 0$.

6. ANOTHER UNIQUENESS THEOREM

Although the hypotheses of Theorem 5.1 include the restriction $\sigma + \sigma^{-1} = 1 + \lambda_*^{-2}$, actually the value σ, by Theorem 2.3, need obey only the condition $\sigma + \sigma^{-1} < 1 + \liminf_m \gamma_m^2 / \xi_m^+ \xi_m^-$. However, no explicit formula gives this limit, whereas (5.5) and (5.6) determine λ_*. Here we strengthen (5.1): we assume that $\lim_m \gamma_{m+1}/\gamma_m$ exists and

$$\mu = \lim_m \gamma_{m+1}/\gamma_m > 0. \tag{6.1}$$

(Again, $\xi_0 \geqslant 0$ and $c > 0$.) Thereby, extending results of Freud [5] and Nevai [9], we obtain a new uniqueness theorem via our initial remark. If $x = (\xi_1, \xi_2, \ldots) \in R^\infty$, then $x(\tau)$ will denote the series $\sum_{m=1}^\infty \xi_m \tau^m$ in the complex variable τ, and $1/s(x)$ will denote the series radius of convergence, whence

$$s(x) = \limsup_m |\xi_m|^{1/m}. \tag{6.2}$$

(the Cauchy–Hadamard formula (Goursat and Hedrick [6, pp. 377–378])). The value $s(x)$, in some sense, measures the "growth rate" of $(\xi_1, \xi_2, \ldots)$. If also $y \in R^\infty$ then $s(x+y) \leqslant \max(s(x), s(y))$, and if $0 \leqslant x \leqslant y$ then $s(x) \leqslant s(y)$.

LEMMA 6.1. (Generalization of Freud [5]). *If $Tx = x$, then ξ_m/γ_m has a limit as $m \to \infty$, and*

$$\theta = \lim_m \xi_m/\gamma_m = (\mu + 1 + \mu^{-1})^{-1/2}. \tag{6.3}$$

Proof. If $\theta_- = \liminf_m \xi_m/\gamma_m$ and $\theta_+ = \limsup_m \xi_m/\gamma_m$, then $0 \leqslant \theta_- \leqslant \theta_+ \leqslant 1$, since $0 < \xi_m/\gamma_m < 1$. Any positive ε determines integer $n(\varepsilon)$ such that

$$\begin{aligned} |(\gamma_{m-1}/\gamma_m) - \mu^{-1}| < \varepsilon, \qquad \xi_m/\gamma_m > \theta_- - \varepsilon, \\ |(\gamma_{m+1}/\gamma_m) - \mu| < \varepsilon, \qquad \xi_m/\gamma_m < \theta_+ + \varepsilon, \end{aligned} \tag{6.4}$$

whenever $m \geqslant n(\varepsilon)$. Accordingly,

$$\begin{aligned} 1 &= \frac{\xi_m}{\gamma_m}\left[\frac{\xi_m}{\gamma_m} + \frac{\gamma_{m+1}}{\gamma_m}\frac{\xi_{m+1}}{\gamma_{m+1}} + \frac{\gamma_{m-1}}{\gamma_m}\frac{\xi_{m-1}}{\gamma_{m-1}}\right] \\ &> \frac{\xi_m}{\gamma_m}\left[\frac{\xi_m}{\gamma_m} + \frac{\mu\xi_{m+1}}{\gamma_{m+1}} + \frac{\xi_{m-1}}{\mu\gamma_{m-1}}\right] - 2\varepsilon \\ &> (\xi_m/\gamma_m)[(\xi_m/\gamma_m) + \theta_-(\mu + \mu^{-1})] - \varepsilon(\mu + 2 + \mu^{-1}), \end{aligned} \tag{6.5}$$

whenever $m > n(\varepsilon)$. If m takes increasing values such that $\xi_m/\gamma_m \to \theta_+$, then

$$1 \geqslant \theta_+[\theta_+ + \theta_-(\mu + \mu^{-1})] - \varepsilon(\mu + 2 + \mu^{-1}), \tag{6.6}$$

and initially ε was arbitrary. Therefore $\theta_+[\theta_+ + \theta_-(\mu + \mu^{-1})] \leqslant 1$; similarly, $1 \leqslant \theta_-[\theta_- + \theta_+(\mu + \mu^{-1})]$. Thus $\theta_+^2 \leqslant \theta_-^2$, or $\theta_- = \theta_+ = \theta$. Finally, $1 = \theta^2[1 + \mu + \mu^{-1}]$.

LEMMA 6.2. *Let* $\omega_- = \liminf_m \omega_m$ *and* $\omega_+ = \limsup_m \omega_m$, where $2 \leqslant \omega_1, \omega_2, \ldots$. *Let $\zeta_0 = 0$ and $z = (\zeta_1, \zeta_2, \ldots) \neq 0$, where the ζ_m satisfy* (2.1). *If σ_- (resp. σ_+) satisfies the relations $\sigma > 1$ and $\sigma + \sigma^{-1} = \omega_-$ (resp. ω_+), then*

$$\sigma_- \leqslant s(z) \leqslant \sigma_+. \tag{6.7}$$

Proof. This simply restates Theorem 2.3.

THEOREM 6.3. (Generalization of Nevai [9]). *If $c = (\gamma_1, \gamma_2, \ldots)$ satisfies* (6.1) *then $Tx = x$ has a unique solution.*

Proof. Recall Theorem 4.3; suppose nonuniqueness. If $\zeta_m = \xi_m^+ - \xi_m^-$, where $m = 0, 1, 2, \ldots$, then $\zeta_0 = 0$ while $\zeta_1, \zeta_2, \ldots > 0$, and the ζ_m obey (2.1) where $\omega_m = 1 + \gamma_m^2/\xi_m^+ \xi_m^-$. Moreover, $\lim_m \omega_m = 1 + \theta^{-2} = \mu + 2 + \mu^{-1}$, by Lemma 6.1, because $x^\pm$ and $x^\mp$ are fixed points of T. If $\sigma > 1$ and

$\sigma+\sigma^{-1}=\mu+2+\mu^{-1}$, then $\mu<\sigma$ by inspection, and $s(z)=\sigma$ by Lemma 6.2. But $0<x^{-}\leqslant x^{+}<c$ and $z=x^{+}-x^{-}$; so $s(z)\leqslant s(c)=\mu$, and this is a contradiction.

Remark. Theorems 4.3, 5.1, 6.3 and Corollary 5.4 provide different uniqueness criteria, and our motivating recurrences (1.3) require no more. If (1.4) had a unique nonnegative solution when $\liminf_m \gamma_{m+1}/\gamma_m<+\infty$, then this test would include all these criteria, but this paper attempts no such comprehensive result. However, the following argument shows that unbounded ratios γ_{m+1}/γ_m permit multiple nonnegative solutions.

Fix $\xi_0\geqslant 0$; then consider any real ξ_1 and use (1.4) to generate $x=(\xi_1,\xi_2,...)$. Either all $\xi_m\neq 0$ and the sequence continues indefinitely, or some $\xi_m=0$ and the sequence terminates there. Moreover, Corollary 3.4 and Lemma 4.2 show that x is a nonnegative solution if and only if $\xi_1\in[\xi_1^-,\xi_1^+]$. If this interval does not contain ξ_1, then x has a first nonpositive element ξ_m. Now equate numerators in (3.8) and equate denominators in (3.8) to define $q_m(\xi)$, $r_m(\xi)$ for all m:

$$\begin{aligned} q_{m+1}&=\gamma_m^2 r_m^2 r_{m-1}-q_m^2 r_{m-1}-q_m r_m q_{m-1},\\ r_{m+1}&=q_m r_m r_{m-1}, \qquad m=1,2,.... \end{aligned} \tag{6.8}$$

Clearly, the factors of each denominator r_m are simply powers of the preceding numerators q_k. Also, the specified ξ_n is the quotient $q_n(\xi_1)/r_n(\xi_1)$. If $\xi_1<\xi_1^+$ (resp. $\xi_1>\xi_1^+$) then $[\xi_1,\xi_1^-]$ (resp. $[\xi_1^+,\xi_1]$) contains a zero of $q_n(\xi)/r_n(\xi)$ unless it contains a pole, and then it contains a zero of some preceding q_m. Hence the zeros of $\{q_m: m=1,2,...\}$ have ξ_1^- and ξ_1^+ as monotone limits. But q_1 has the unique root 0, while q_2 has the positive root $\gamma_1 g(\xi_0/2\gamma_1)$. Given any positive ε, choose all γ_m^2 so large that no root of q_{m+1}, via (6.8), exceeds distance $\varepsilon/2^{m+1}$ from some root of $r_m^2 r_{m-1}$. The roots of the latter are all roots of some preceding q_k. Therefore, $\xi_1^-\neq\xi_1^+$ when ε is small enough.

7. Sequence Computation

The recurrence (1.3), by definition, has fixed value $\xi_0=0$, and Section 8 will show that this recurrence, by Theorem 4.3, has unique nonnegative solution $x^*=(\xi_1^*,\ \xi_2^*,...)$; also, independent remarks will furnish ξ_1^*. Theoretically, ξ_0 and ξ_1^*, via (1.3), determine all ξ_m^* for higher m. However, we find here that such forward iteration is an unstable algorithm, and we give a stable one. Hereafter, we fix values $\gamma_0=\xi_0=0$ because they simplify the discussion, and we take any positive sequence $c=(\gamma_1,\gamma_2,...)$ such that (1.4) has unique nonnegative solution x^*. (Later results will need assumption

(6.1).) Further, we choose any $\xi_1 > 0$ and, from ξ_0, ξ_1, we define $x = (\xi_1, \xi_2, ...)$ via (1.4). This sequence x represents a hypothetical computation. Clearly, a program can set $\xi_0 = 0$, but it cannot set $\xi_1 = \xi_1^*$ unless the latter is a machine number—and example (1.3) will have irrational ξ_1^*. Moreover, if the machine ξ_1 were actually ξ_1^*, still, roundoff error would soon produce an m such that the computed ξ_m was not ξ_m^*; so the effective ξ_1 would not be precisely ξ_1^*.

However, *any* distinct ξ_1 and ξ_1^* yield increasingly different ξ_m and ξ_m^*. If $\xi_1 \neq \xi_1^*$ then $x \neq x^*$, and either some finite $\xi_m = 0$, whence x has no further elements, or else some finite $\xi_m < 0$, since x^* is the *only* nonnegative solution. If ξ_{n+2} is the first nonpositive element of x, then $0 < \xi_1, ..., \xi_{n+1}$, and $\xi_m \leqslant \gamma_m$, by Lemma 3.1, when $1 \leqslant m \leqslant n$. Now let $(\gamma_1, \gamma_2, ...)$ satisfy (5.1), whence (5.4) defines constants κ_*, λ_*; and let $\zeta_m = (-1)^m(\xi_m - \xi_m^*)$ for $m = 0, 1, 2, ...$. Then $\zeta_0 = 0$ and $\zeta_1 \neq 0$, while repeating calculation (4.11) produces again recurrence (2.1), where

$$\omega_m = 1 + \gamma_m^2/\xi_m^*\xi_m \geqslant 1 + \lambda_*^{-1} \cdot \min\{\gamma_m/\xi_m : m = 1, ..., n\} \geqslant 1 + \lambda_*^{-1} > 2 \quad (7.1)$$

for $m = 1, ..., n$. But using Lemma 2.2 gives immediately

$$\zeta_{m+1}/\zeta_1 = C_m(\omega_1, ..., \omega_m) \geqslant C_m(1 + \lambda_*^{-1} \cdot \min\{\gamma_m/\xi_m : m = 1, ..., n\}) \quad (7.2)$$

for $m = 1, ..., n$. Therefore, the discrepancy $\xi_m - \xi_m^*$ has alternating sign and exponentially growing magnitude until the value ξ_m becomes zero or a negative quantity. Clearly, (1.4) has the same instability for backward iteration, because (2.1) has the same properties for decreasing m.

Since we cannot obtain $\xi_1^*, ..., \xi_n^*$ by an initial-value method, instead we shall compute these numbers via a boundary-value approach. Hereafter, we assume (6.1), and invoking Lemma 6.1, initially we use (6.3) to make the estimates

$$\xi_m = (\mu + 1 + \mu^{-1})^{-1/2}\gamma_m, \qquad m = 1, 2, ..., n+1. \quad (7.3)$$

Now, fixing this value ξ_{n+1}, we solve (3.1) or, recalling the equivalent problem, we minimize $F(\xi_1, ..., \xi_m)$. However, this is a trivial matter unless $n > 1$; and then, by Lemma 3.1, the domain K^0 contains the minimum, where

$$K^0 = \{x \in R^n : 0 < \xi_m < \gamma_m, m = 1, ..., n\}, \quad (7.4)$$

while also, on this domain, Theorem 3.2 shows that $F(x)$ is an analytic, strictly convex function. The literature contains many optimization methods, but these facts suggest a Newton–Raphson algorithm. Specifically, taking (7.3) as a first guess, we solve iteratively $F'(x) = 0$. Indeed, given any point x_{old} in K^0, we compute a successor $x_{\text{new}} = x_{\text{old}} - \Delta x$, where (tentatively)

$$\Delta x \cdot F''(x_{\text{old}}) = F'(x_{\text{old}}) \quad (7.5)$$

provided K^0 contains the resulting x_{new}; otherwise some $\alpha \cdot \Delta x$ replaces this Δx, where $0 < \alpha < 1$. The $n \times n$ Hessian matrix $F''(x)$, on the domain K^0, is symmetric, tridiagonal, and diagonally dominant by (3.6). Also, the subdiagonal has all entries unity. Hence the system (7.5), through Gaussian elimination, produces the required Δx in $2n - 1$ multiplications/divisions.

Still, our goal is the values ξ_m^*. If the estimated ξ_{n+1}, via this algorithm, determines $\xi_1,..., \xi_n$ with high precision, then the remaining error in $\xi_1,..., \xi_n$ reflects the initial error in ξ_{n+1}. Indeed, if $\sigma > 1$ and $\sigma + \sigma^{-1} \leqslant \min(\omega_1,..., \omega_n)$, hence if

$$\sigma + \sigma^{-1} \leqslant 1 + \lambda_*^{-1} \cdot \min\{\gamma_m/\xi_m : m = 1,..., n\} \tag{7.6}$$

and $\zeta_1 \neq 0$, then $\zeta_{k-1}/\zeta_k < 1/\sigma$ for $k = 1,..., n+1$. Obviously this is true when $k = 1$, and if it is true when $1 \leqslant k \leqslant m$, then

$$\zeta_{m+1}/\zeta_m = \omega_m - (\zeta_{m-1}/\zeta_m) \geqslant \sigma + \sigma^{-1} - (\zeta_{m-1}/\zeta_m) > \sigma. \tag{7.7}$$

Thus $\zeta_m/\zeta_{n+1} < \sigma^{m-n-1}$, or

$$|\xi_m - \xi_m^*| < |\xi_{n+1} - \xi_{n+1}^*| \cdot \sigma^{m-n-1}. \tag{7.8}$$

Even the poor estimate (7.3) for ξ_{n+1} should therefore yield accurate values for earlier ξ_m. Roughly bounding the error $|\xi_{n+1} - \xi_{n+1}^*|$, we merely choose extra large n and keep only those ξ_m where (7.8) implies small enough error $|\xi_m - \xi_m^*|$.

Here $\xi_{n+1} = (\mu + 1 + \mu^{-1})^{-1/2} \gamma_{n+1}$ by (7.3), while $\kappa_* \gamma_{n+1} \leqslant \xi_{n+1}^* \leqslant \lambda_* \gamma_{n+1}$ by (5.9). Thus

$$\begin{aligned} |\xi_{n+1} - \xi_{n+1}^*| \leqslant \gamma_{n+1} \cdot \max(&|\kappa_* - (\mu + 1 + \mu^{-1})^{-1/2}|, \\ &|\lambda_* - (\mu + 1 + \mu^{-1})^{-1/2}|). \end{aligned} \tag{7.9}$$

Further ingenuity may improve this bound. If we can find $y^- = (\eta_1^-, \eta_2^-,...) \in R^\infty$ and $y^+ = (\eta_1^+, \eta_2^+,...) \in R^\infty$ such that

$$y^- \leqslant Ty^+ \leqslant Ty^- \leqslant y^+, \tag{7.10}$$

then we can conclude $T([y^-, y^+]) \subset [y^-, y^+]$, hence $x^* \in [y^-, y^+]$, so that

$$\eta_{n+1}^- \leqslant \xi_{n+1}^* \leqslant \eta_{n+1}^+. \tag{7.11}$$

Indeed, $x^* \in [T^k y^-, T^k y^+]$ for $k = 0, 1, 2,...$, but iterating T may require too much computation. Section 9 uses this device.

8. Freud's Recurrence

Recurrence (1.3) involves a parameter ρ such that $\rho > -1$. Specifically, $\xi_0 = 0$ and $c = c(\rho) = (\gamma_1(\rho), \gamma_2(\rho),...) > 0$, where

$$\gamma_m(\rho)^2 = (\tfrac{1}{8})[2m + \rho - \rho(-1)^m] \qquad \text{for } m = 1, 2,.... \tag{8.1}$$

Putting $m = 0$ yields $\gamma_0(\rho) = 0$, and this is an appropriate value since it obeys relations (5.7). Also, $\lim_m \gamma_m(\rho)/m = 0$ and $\lim_m \gamma_{m+1}(\rho)/\gamma_m(\rho) = 1$. Hence $0 < \phi_{c(\rho)} \leqslant \psi_{c(\rho)} < +\infty$ and $0 < \kappa_*(\rho) \leqslant \lambda_*(\rho) < 1$. Thus (1.3) has a nonnegative solution x^* by Theorem 4.3, and x^* is the *only* nonnegative solution by Theorem 4.3 or 6.3. Moreover, any two sequences $c(\rho_1)$, $c(\rho_2)$ define equivalent norms $\|\cdot\|_{c(\rho)}$; and obviously $c(0)$, in particular, is a concave sequence. Therefore, if $y \in R^\infty$ then $\lim_k \|T^k y - x^*\|_{c(\rho)} = 0$ by Corollary 5.4; and if $x^* = (\xi_1^*, \xi_2^*,...)$ then $\lim_m \xi_m^*/\gamma_m(\rho) = 1/\sqrt{3}$ by Lemma 6.1.

Indeed, we can refine the last statement. Putting $\tau = 1/m$, suppose

$$\begin{aligned} (12/m)^{1/2}\, \xi_m^* &= f^+(\tau) = 1 + \alpha_1^+ \tau + \alpha_2^+ \tau^2 + ..., \qquad && m \text{ even}, \\ (12/m)^{1/2}\, \xi_m^* &= f^-(\tau) = 1 + \alpha_1^- \tau + \alpha_2^- \tau^2 + ..., && m \text{ odd}. \end{aligned} \tag{8.2}$$

Then (1.3) implies the relations

$$\begin{aligned} 0 = (-3/f^+(\tau)) + f^+(\tau) + (1-\tau)^{1/2} f^-(\tau/(1-\tau)) \\ + (1+\tau)^{1/2} f^-(\tau/(1+\tau)), \end{aligned} \tag{8.3}$$

$$\begin{aligned} 0 = (1+\rho\tau)(-3/f^-(\tau)) + f^-(\tau) + (1-\tau)^{1/2} f^+(\tau/(1-\tau)) \\ + (1+\tau)^{1/2} f^+(\tau/(1+\tau)). \end{aligned} \tag{8.4}$$

The coefficient of each power τ^k in Eq. (8.3) (resp. Eq. (8.4)) has the form $4\alpha_k^+ + 2\alpha_k^-$ (resp. $4\alpha_k^- + 2\alpha_k^+$) plus some terms in the prior α_j^+, α_j^-. Therefore, equating these coefficients to zero yields inductively the values of all α_k^+, α_k^-. Computing the first few produces the following series:

$$\begin{aligned} (12/m)^{1/2}\, \xi_m^* = 1 &+ \tau[-\rho/2] + \tau^2[(\rho^2/4) + (1/24)] \\ &+ \tau^3[(-\rho^3/16) + (-3\rho/8)] \\ &+ \tau^4[(-5\rho^4/128) + (35\rho^2/64) + (-7/576)] + ..., \qquad m \text{ even}; \end{aligned} \tag{8.5}$$

$$(12/m)^{1/2}\,\xi_m^* = 1 + \tau[\rho] + \tau^2[(-\rho^2/8) + (1/24)] + \tau^3[(-\rho^3/16) + (5\rho/16)] + \tau^4[(5\rho^4/64) + (-5\rho^2/16) + (-7/576)] + \dots, \qquad m \text{ odd.} \tag{8.6}$$

So far, these expansions have only formal significance, but we show next that (8.5), (8.6) are asymptotic series. If $x = (\xi_1, \xi_2, \dots) \in R^\infty$ and $0 < d = (\delta_1, \delta_2, \dots) \in R^\infty$, then $\xi_m = O(\delta_m)$ as $m \to \infty$ if and only if $\sup\{|\xi_m/\delta_m| : m = 1, 2, \dots\} < +\infty$ (Erdélyi [2, p. 5]), or $\|x\|_d < +\infty$. In particular, if $\delta_m = m^\nu$ for $m > n$, where ν is any real number and n is any positive integer, then $\xi_m = O(m^\nu)$ as $m \to \infty$.

LEMMA 8.1. *Any real $\rho > -1$ and real $\nu \leqslant 0$ determine an integer $n > 0$ and a sequence $d(\nu) = (\delta_1, \delta_2, \dots) > 0$ such that $\kappa_* \delta_0 \leqslant \xi_0 \leqslant \lambda_* \delta_0$,* where $\delta_0 = 0$, while

$$\psi_{d(\nu)}\,|g'(\kappa_*(\rho)\,\phi_{c(\rho)})| < 1, \tag{8.7}$$

where $\psi_{d(\nu)} = \sup\{(\delta_{m-1} + \delta_{m+1})/2\delta_m : m = 1, 2, \dots\}$, and $\delta_m = m^\nu$ whenever $m > n$.

Proof. If $\delta_0 = 0$ and otherwise $\delta_m = m^\nu$ then clearly

$$(\delta_{m-1} + \delta_{m+1})/2\delta_m = (\tfrac{1}{2})[(1 - m^{-1})^\nu + (1 + m^{-1})^\nu] \to 1 \tag{8.8}$$

as $m \to \infty$. But $|g'(\kappa_*(\rho)\,\phi_{c(\rho)})| < 1$; hence (8.8) implies some integer n such that

$$|g'(\kappa_*(\rho)\,\phi_{c(\rho)})| \cdot (\delta_{m-1} + \delta_{m+1})/2\delta_m < 1 \tag{8.9}$$

when $m > n$. If we define $\delta_m = n^\nu$ when $1 \leqslant m \leqslant n$, then we satisfy (8.9) when $m = 1, 2, \dots$. The first condition is trivial because $\xi_0 = 0$.

THEOREM 8.2. *Expansions* (8.5), (8.6) *are asymptotic series.*

Proof. If we truncate (8.5), (8.6) at the power τ^k then the resulting polynomials will be nonnegative for small positive τ, and we can add multiples of the power τ^{k+1} so that the resulting $g^+(\tau)$, $g^-(\tau)$ will be nonnegative for *all* positive τ. If $h^+(\tau) \geqslant 0$, $h^-(\tau) \geqslant 0$, where

$$\begin{aligned} 0 &= (-3/h^+(\tau)) + h^+(\tau) + (1-\tau)^{1/2}\,g^-(\tau/(1-\tau)) \\ &\quad + (1+\tau)^{1/2}\,g^-(\tau/(1+\tau)), \\ 0 &= (1+\rho\tau)(-3/h^-(\tau)) + h^-(\tau) + (1-\tau)^{1/2}\,g^+(\tau/(1-\tau)) \\ &\quad + (1+\tau)^{1/2}\,g^+(\tau/(1+\tau)), \end{aligned} \tag{8.10}$$

then $h^+(\tau)$ and $h^-(\tau)$ have convergent power series for small enough $|\tau|$, while $g^+(\tau)$ and $h^+(\tau)$, $g^-(\tau)$ and $h^-(\tau)$, have the *same* such expansions up to powers τ^k. If $x=(\xi_1, \xi_2,...)$, where $\xi_m=(m/12)^{1/2} g^+(1/m)$ (resp. $(m/12)^{1/2} g^-(1/m)$) when m is even (resp. odd), then $x \geqslant 0$ and $(Tx)_m=(m/12)^{1/2} h^+(1/m)$ (resp. $(m/12)^{1/2} h^-(1/m)$) when m is even (resp. odd). Here we exclude components with $m=1$. Hence $|(Tx)_m-\xi_m|= O(m^{-k-(1/2)})$ as $m\to\infty$. If $\nu=-k-(1/2)$ in Lemma 8.1, then $\|Tx-x\|_{d(\nu)}<+\infty$ for the corresponding $d(\nu)$. But T has a unique fixed point x^* by Theorem 4.3, and $\|x^*-x\|_{d(\nu)}<+\infty$ by Theorem 5.3. Thus $\xi^*_m-\xi_m=O(m^{-k-(1/2)})$ as $m\to\infty$.

9. Computational Refinements

Freud's [3–5] papers assume the weight function $w(\tau)=|\tau|^\rho \exp(-|\tau|^r)$, and this function is clearly even; so each orthonormal polynomial $p_m(\tau)$ is either even or odd. Indeed, $p_0(\tau)=\pi_0$ and $p_1(\tau)=\pi_1\tau$, where π_0, π_1 are constants and our definition makes them positive. Then the orthonormality relations (1.1) and standard gamma-function integrals (Abramowitz and Stegun [1, Eq. (6.1.1)]) yield easily

$$r/2\pi_0^2=\Gamma((\rho+1)/r), \qquad r/2\pi_1^2=\Gamma((\rho+3)/r). \tag{9.1}$$

But Section 1 fixes $r=4$, and system (1.3) assumes $\xi_m=(\pi_{m-1}/\pi_m)^2$. Hence the values $(\pi_{m-1}/\pi_m)^2$ from the polynomials p_m form the unique nonnegative solution x^* of Freud's recurrence, and

$$\xi_0^*=0, \qquad \xi_1^*=\Gamma((\rho+3)/4)/\Gamma((\rho+1)/4). \tag{9.2}$$

Theoretically, the recurrence now determines all further ξ^*_m; but (7.2) shows that such forward iteration is an exponentially unstable algorithm, and experiment confirms this. However, the boundary-value approach of Section 7 yields accurately the values ξ^*_m for larger m.

Moreover, Section 8 motivates a further refinement. To obtain an initial estimate for the computed ξ_m, truncate the series (8.5)–(8.6) at some power τ^k. Then $|\xi_m-\xi^*_m|\leqslant \text{constant}\cdot\tau^{k+(1/2)}$, where $\tau=1/m$; so this bound is worst when m is smallest. Let $\xi^*_1,..., \xi^*_n$ be the desired values, and $\varepsilon_m=\gamma\cdot m^{-k-(1/2)}$, taking γ as the constant such that ε_n is the desired tolerance. Let $n(j)=\min(n, 2^j)$ where $j=0, 1, 2,...$. Then the j'th step of our refined algorithm accepts $\xi_{n(j)+1},..., \xi_{n+1}$ from the truncated series, but it uses Newton iteration to improve $\xi_1,..., \xi_{n(j)}$ until the tolerance $\varepsilon_{n(j)}$ for this step exceeds the absolute maximum of the first $n(j)$ residuals. Each step, with increasing precision, clearly adjusts those values ξ_m which, on the next step, would otherwise need the most correction. Thus each step computes a longer sequence until $n(j)$ reaches n, but early steps involve quite small linear systems (7.5).

If $\rho = 0$ then $\gamma_m = \sqrt{m}/2$ by (8.1), whence $\phi_c = 1/\sqrt{2}$ and $\psi_c = 1$ by Section 5; so

$$\kappa_* \simeq 0.52200941, \qquad \lambda_* \simeq 0.69683244 \tag{9.3}$$

by (5.5)–(5.6). Moreover, symbolic computation generates the series (Trager [11]):

$$\begin{aligned} f^+(\tau) = f^-(\tau) = 1 &+ \tau^2/2^3 \cdot 3 - 7\tau^4/2^6 \cdot 3^2 + 37\tau^6/2^{10} \cdot 3^2 \\ &+ 92{,}963\tau^8/2^{15} \cdot 3^4 - 200{,}039\tau^{10}/2^{16} \cdot 3^2 + 7{,}394{,}856{,}055\tau^{12}/2^{22} \cdot 3^6 \\ &- 416{,}852{,}554{,}595\tau^{14}/2^{25} \cdot 3^7 + \dots . \end{aligned} \tag{9.4}$$

Indeed, one can prove inductively that (9.4) contains no odd powers. The structure of Eq. (8.3) and (8.4) implies the nonconvergence of this series, which therefore shows that one cannot strengthen Theorem 8.2. The authors used several terms of this series for $\rho = 0$, and they tried the refined algorithm of this section with $n = 250$. No step needed more than one Newton iteration; some steps required none at all. Thus the numerical linear algebra did not exceed $4n$ multiplications/divisions. To improve further the resulting $\xi_1, \dots, \xi_m$, the authors took these computed values and applied repeatedly the map $U: x \to (x + Tx)/2$, that is,

$$(Ux)_m = (\xi_m + (Tx)_m)/2, \qquad m = 1, \dots, n. \tag{9.5}$$

This process has much slower convergence, but its starting-point was almost the exact solution. Hence the numerical values, after six iterations, showed complete stability in 64-bit floating-point arithmetic. Appendix A gives the first 20 terms.

Having thus minimized roundoff, we ignore it, and, to estimate accuracy, we use (7.8). To find σ we need (7.6). For a better value we could explicitly substitute $\min\{\gamma_m/\xi_m : m = 1, \dots, n\}$, but for a faster result we need verify only that $\xi_m \leqslant \lambda_* \gamma_m$ when $1 \leqslant m \leqslant n$. A conservative σ then satisfies the equation $\sigma + \sigma^{-1} = 1 + \lambda_*^{-2}$, whence $\sigma \simeq 2.6872901$ and $\sigma^7 \simeq 1012.0498 > 10^3$; so dropping seven terms from the end of the computed sequence improves the accuracy by a factor of at least 10^3. Even (7.9) yields the crude estimate that $|\xi_{251} - \xi_{251}^*| < 1$, whence $|\xi_m - \xi_m^*| < 10^{-15}$ for $m \leqslant 216$; but (7.11) gives a better bound when

$$\eta_m^- = ((2m-1)/24)^{1/2}, \quad \eta_m^+ = ((2m+1)/24)^{1/2}, \qquad m = 1, 2, \dots . \tag{9.6}$$

(Direct computation checks (7.10) for $m = 1$; simple algebra proves it for $m > 1$.) Now $|\xi_{251} - \xi^*_{251}| < 0.00456$ because ξ^*_{251} satisfies (7.11), and $|\xi_m - \xi^*_m| \leqslant 2 \times 10^{-15}$ when $m \leqslant 222$. Indeed, if ξ_{n+1} is the last value, for any $n \geqslant 29$, while $\eta^-_{n+1} \leqslant \xi_{n+1} \leqslant \eta^+_{n+1}$, then

$$|\xi_{n+1} - \xi^*_{n+1}| \leqslant |\eta^+_{n+1} - \eta^-_{n+1}| \leqslant |\eta^+_{30} - \eta^-_{30}| < 0.0264, \tag{9.7}$$

so that $|\xi_m - \xi^*_m| < 10^{-14}$ for any $m \leqslant n - 28$.

Another argument improves this estimate. If $\delta_0 = 0$ and sequence $d = (\delta_1, \delta_2, ...)$, where $\delta_1 = \cdots = \delta_9 = 1/27$ and otherwise $\delta_m = m^{-3/2}$, then Sections 4 and 5 show that

$$\alpha = \psi_d\, |g'(\kappa_* \phi_c)| \leqslant 2/3. \tag{9.8}$$

If $y = (\eta_1, \eta_2, ...) \in R^\infty$, where $\eta_1, ..., \eta_{251}$ are the computed $\xi_1, ..., \xi_{251}$ and otherwise

$$\eta_m = (m/12)^{1/2}[1 + (1/24m^2) - (7/576m^4)], \tag{9.9}$$

then $\|Ty - y\|_d < +\infty$. Also, $\kappa_* c \leqslant y$ by inspection, whence $\|T^k y - x^*\|_d \to 0$ by Theorem 5.3. But Lemma 5.2 implies that

$$\begin{aligned} \|y - x^*\|_d &\leqslant \sum_{k=0}^{q-1} \|T^{k+1}y - T^k y\|_d + \|T^q y - x^*\|_d \\ &\leqslant \sum_{k=0}^{\infty} \alpha^k \|Ty - y\|_d \leqslant 3\, \|Ty - y\|_d; \end{aligned} \tag{9.10}$$

whence $|\xi_m - \xi^*_m| \leqslant 3 \|Ty - y\|_d \cdot m^{-3/2}$ for $1 \leqslant m \leqslant 251$. A finite computation yields $\|Ty - y\|_d$.

Clearly, our ξ_m should be very accurate when the index m is small. If we put $\rho = 0$ and use (9.2) to find ξ^*_1, then ξ_1, from the preceding algorithm, has the same value to the last computed digit.

Appendix A

If $\rho = 0$ and $\xi_0 = 0$, then the arguments of Section 8 show that system (1.3) has unique nonnegative solution x^*, and the method of Section 9 finds that components $\xi^*_1, ..., \xi^*_{20}$ have the following values.

m	ξ_m^*	m	ξ_m^*
1	0.33798912003364232	11	0.95775608488417893
2	0.40167965976351733	12	1.0002887465597798
3	0.50510423234482221	13	1.0410891789282268
4	0.57805815033171129	14	1.0803527252385585
5	0.64676738204724493	15	1.1182407644978825
6	0.70786315090515241	16	1.1548882639750164
7	0.76442312605207728	17	1.1904095014212202
8	0.81702175201098198	18	1.2249022329454762
9	0.86647036419002228	19	1.2584508557944947
10	0.91324989944000748	20	1.2911288293490708

Appendix B

If all γ_m have the same value γ, then truncations of (1.5) yield solutions of (3.1) where the ξ_m have varying sign. This appendix, for equal γ_m, lists more solutions of this finite system. If any real $\gamma_1, \dots, \gamma_n$ and $\xi_0, \dots, \xi_{n+1}$ satisfy (3.1), then the constant multiples $\alpha\gamma_1, \dots, \alpha\gamma_n$ and $\alpha\xi_0, \dots, \alpha\xi_{n+1}$ satisfy (3.1). Hence, taking $\gamma_1 = \cdots = \gamma_n = 1$ and fixing $\xi_0 = 1 = \pm\xi_{n+1}$, we give *all* real solutions for the smallest few n. We suppress the algebraic details for brevity; the final result suggests the intermediate substitutions.

If $n = 1$ and $\xi_2 = -1$, then $(\xi_0, \xi_1, \xi_2) = (1, \eta, -1)$, where $\eta = \pm 1$. If $n = 2$ and $\xi_3 = -1$, then

$$(\xi_0, \dots, \xi_3) = (1, 1, -1, -1) \qquad \text{or} \qquad (\tfrac{1}{2})(2, \eta - 1, \eta + 1, -2), \tag{B.1}$$

where $\eta = \pm\sqrt{3}$. If $n = 3$ and $\xi_4 = 1$, then

$$\begin{aligned}(\xi_0, \dots, \xi_4) = (1, 1, -1, -1, 1) \qquad &\text{or} \qquad (1, -1, -1, 1, 1) \\ &\text{or} \qquad (\tfrac{1}{2})(2, \eta - \eta^{-1}, 2\eta^{-1}, \eta - \eta^{-1}, 2),\end{aligned} \tag{B.2}$$

where $\eta^4 + 2\eta^3 - 4\eta^2 - 2\eta - 1 = 0$. This last equation has just two real roots:

$$\eta_1 \simeq 1.5815460, \qquad \eta_2 \simeq -3.1120097. \tag{B.3}$$

If $n = 4$ and $\xi_5 = 1$, then

$$\begin{aligned}(\xi_0, \dots, \xi_5) = (1, 1, -1, -1, 1, 1) & \\ \text{or} \qquad & (1, \eta^{-1}, -1 + \eta, -1 - \eta, -\eta^{-1}, 1) \\ \text{or} \qquad & (1, \zeta - 2\zeta^{-1}, \zeta^{-1}, \zeta^{-1}, \zeta - 2\zeta^{-1}, 1),\end{aligned} \tag{B.4}$$

where $\eta = \pm 1/\sqrt{2}$ and $\zeta^3 - 4\zeta + 2 = 0$. This last equation has three real roots:

$$\zeta_1 \simeq 1.6751309, \qquad \zeta_2 \simeq 0.53918887, \qquad \zeta_3 \simeq -2.2143197. \qquad \text{(B.5)}$$

No two sequences with the same n have the same pattern of signs. However, it is unclear whether this is significant.

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